# Dynamic programming

- memoization
- decorator memoized / functools.cache
- systematic subproblem computation

# Binomial coefficient

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{otherwise} \end{cases}$$

**binomial_recursive.py**

```python
def binomial(n, k):
    if k == 0 or k == n:
        return 1
    return binomial(n - 1, k) + binomial(n - 1, k - 1)
```

```
--(7, 5)
   |--(6, 5)
   |   |--(5, 5)
   |    --(5, 4)
   |       |--(4, 4)
   |        --(4, 3)
   |           |--(3, 3)
   |            --(3, 2)
   |               |--(2, 2)
   |                --(2, 1)
   |                   |--(1, 1)
   |                    --(1, 0)
    --(6, 4)
       |--(5, 4)
       |   |--(4, 4)
       |    --(4, 3)
       |       |--(3, 3)
       |        --(3, 2)
       |           |--(2, 2)
       |            --(2, 1)
       |               |--(1, 1)
       |                --(1, 0)
        --(5, 3)
           |--(4, 3)
           |   |--(3, 3)
           |    --(3, 2)
           |       |--(2, 2)
           |        --(2, 1)
           |           |--(1, 1)
           |            --(1, 0)
            --(4, 2)
               |--(3, 2)
               |   |--(2, 2)
               |    --(2, 1)
               |       |--(1, 1)
               |        --(1, 0)
                --(3, 1)
                   |--(2, 1)
                   |   |--(1, 1)
                   |    --(1, 0)
                    --(2, 0)
```

identical computations

recursion tree for `binomial(7,5)`

Dynamic Programming

≡

Remember solutions already found

(memoization)

- Technique sometimes applicable when running time otherwise becomes exponential
- Only applicable if stuff to be remembered is manageable

# Binomial Coefficient
# Dynamic programming using a dictionary

recursion tree for `binomial(7,5)`

```
--(7, 5)
  |--(6, 5)
  |   |--(5, 5)
  |    --(5, 4)
  |       |--(4, 4)
  |        --(4, 3)
  |           |--(3, 3)
  |            --(3, 2)
  |               |--(2, 2)
  |                --(2, 1)
  |                   |--(1, 1)
  |                    --(1, 0)
   --(6, 4)
      |--(5, 4)
       --(5, 3)
          |--(4, 3)
           --(4, 2)
              |--(3, 2)
               --(3, 1)
                  |--(2, 1)
                   --(2, 0)
```

reuse value stored in dictionary `answers`

**binomial_dictionary.py**

```python
answers = {}  # answers[(n, k)] = binomial(n, k)

def binomial(n, k):
    if (n, k) not in answers:
        if k == 0 or k == n:
            answer = 1
        else:
            answer = binomial(n - 1, k) + binomial(n - 1, k - 1)
        answers[(n, k)] = answer
    return answers[(n, k)]
```

**Python shell**

```
> binomial(6, 3)
| 20
> answers
| {(3, 3): 1, (2, 2): 1, (1, 1): 1, (1, 0): 1, (2, 1): 2, (3, 2):
  3, (4, 3): 4, (2, 0): 1, (3, 1): 3, (4, 2): 6, (5, 3): 10, (3,
  0): 1, (4, 1): 4, (5, 2): 10, (6, 3): 20}
```

- Use a dictionary `answers` to store already computed values

# Question – What is the order of the size of the dictionary `answers` after calling `binomial(n, k)` ?

**binomial_dictionary.py**

```
answers = {}  # answers[(n, k)] = binomial(n, k)

def binomial(n, k):
    if (n, k) not in answers:
        if k == 0 or k == n:
            answer = 1
        else:
            answer = binomial(n - 1, k) + binomial(n - 1, k - 1)
        answers[(n, k)] = answer
    return answers[(n, k)]
```

a) max(n, k)
b) n + k
c) n * k
d) $n^k$
e) $k^n$
f) Don't know

# Binomial Coefficient
# Dynamic programming using decorator

- Use a decorator (@memoize) that implements the functionality of remembering the results of previous function calls

**binomial_decorator.py**

```
def memoize(f):
    # answers[args] = f(*args)
    answers = {}

    def wrapper(*args):
        if args not in answers:
            answers[args] = f(*args)
        return answers[args]

    return wrapper
```

```
@memoize
def binomial(n, k):
    if k == 0 or k == n:
        return 1
    else:
        return binomial(n - 1, k) + binomial(n - 1, k - 1)
```

www.python-course.eu/python3_memoization.php

## binomial_decorator_trace.py

```python
def trace(f):  # decorator to trace recursive calls
    indent = 0

    def wrapper(*args):
        nonlocal indent

        spaces = '|  ' * indent
        arg_str = ', '.join(map(repr, args))
        print(spaces + f'{f.__name__}({arg_str})')
        indent += 1
        result = f(*args)
        indent -= 1
        print(spaces + f'> {result}')
        return result

    return wrapper

def memoize(f):
    answers = {}

    def wrapper(*args):
        if args not in answers:
            answers[args] = f(*args)
        return answers[args]

    wrapper.__name__ = f.__name__ + '_memoize'

    return wrapper

@trace
@memoize
def binomial(n, k):
    if k == 0 or k == n:
        return 1
    return binomial(n - 1, k) + binomial(n-1, k-1)

print(binomial(5, 2))
```

## Python shell (without @memoize)

```
binomial(5, 2)
|  binomial(4, 2)
|  |  binomial(3, 2)
|  |  |  binomial(2, 2)
|  |  |  > 1
|  |  |  binomial(2, 1)
|  |  |  |  binomial(1, 1)
|  |  |  |  > 1
|  |  |  |  binomial(1, 0)
|  |  |  |  > 1
|  |  |  > 2
|  |  > 3
|  |  binomial(3, 1)
|  |  |  binomial(2, 1)
|  |  |  |  binomial(1, 1)
|  |  |  |  > 1
|  |  |  |  binomial(1, 0)
|  |  |  |  > 1
|  |  |  > 2
|  |  |  binomial(2, 0)
|  |  |  > 1
|  |  > 3
|  > 6
|  binomial(4, 1)
|  |  binomial(3, 1)
|  |  |  binomial(2, 1)
|  |  |  |  binomial(1, 1)
|  |  |  |  > 1
|  |  |  |  binomial(1, 0)
|  |  |  |  > 1
|  |  |  > 2
|  |  |  binomial(2, 0)
|  |  |  > 1
|  |  > 3
|  |  binomial(3, 0)
|  |  > 1
|  > 4
> 10
10
```

saved recursive calls
when using memoization

## Python shell (with @memoize)

```
binomial_memoize(5, 2)
|  binomial_memoize(4, 2)
|  |  binomial_memoize(3, 2)
|  |  |  binomial_memoize(2, 2)
|  |  |  > 1
|  |  |  binomial_memoize(2, 1)
|  |  |  |  binomial_memoize(1, 1)
|  |  |  |  > 1
|  |  |  |  binomial_memoize(1, 0)
|  |  |  |  > 1
|  |  |  > 2
|  |  > 3
|  |  binomial_memoize(3, 1)
|  |  |  binomial_memoize(2, 1)
|  |  |  > 2
|  |  |  binomial_memoize(2, 0)
|  |  |  > 1
|  |  > 3
|  > 6
|  binomial_memoize(4, 1)
|  |  binomial_memoize(3, 1)
|  |  > 3
|  |  binomial_memoize(3, 0)
|  |  > 1
|  > 4
> 10
10
```

without assigning `wrapper.__name__`
the name shown would be `wrapper`

# Dynamic programming using cache decorator

**bionomial_cache.py**

```
from functools import cache

@cache
def binomial(n, k):
    if k == 0 or k == n:
        return 1
    else:
        return binomial(n - 1, k) + binomial(n - 1, k - 1)
```

- The decorators `@cache` (since Python 3.9) and `@lru_cache(maxsize=None)` in the standard library `functools` supports the same as the decorator `@memoize`
- By default `@lru_cache` at most remembers (caches) 128 previous function calls, always evicting **L**east **R**ecently **U**sed entries from its dictionary

docs.python.org/3/library/functools.html#functools.lru_cache

# Subset sum using dynamic programming

- In the subset sum problem (Exercise 13.4) we are given a number `x` and a list of numbers `L`, and want to determine if a subset of `L` has sum `x`

```
L = [3, 7, 2, 11, 13, 4, 8]    x = 22 = 7 + 11 + 4
```

- Let S(v, k) denote if it is possible to achieve value v with a subset of `L[:k]`, i.e. S(v, k) = True if and only if a subset of the first k values in `L` has sum v
- S(v, k) can be computed from the recurrence

$$S(v,k) = \begin{cases} \text{True} & \text{if } k = 0 \text{ and } v = 0 \\ \text{False} & \text{if } k = 0 \text{ and } v \neq 0 \\ S(v, k-1) \text{ or } S(v - L[k-1], k-1) & \text{otherwise} \end{cases}$$

# Subset sum using dynamic programming

**subset_sum_dp.py**

```python
def subset_sum(x, L):
    @memoize
    def solve(value, k):
        if k == 0:
            return value == 0
        return solve(value, k - 1) or solve(value - L[k - 1], k - 1)
    return solve(x, len(L))
```

**Python shell**

```
> subset_sum(11, [2, 3, 8, 11, -1])
| True
> subset_sum(6, [2, 3, 8, 11, -1])
| False
```

# Question – What is a bound on the size order of the memoization table if all values are possitive integers?

**subset_sum_dp.py**

```
def subset_sum(x, L):
    @memoize
    def solve(value, k):
        if k == 0:
            return value == 0
        return solve(value, k-1) or solve(value - L[k-1], k-1)
    return solve(x, len(L))
```

**Python shell**

```
> subset_sum(11, [2, 3, 8, 11, -1])
| True
> subset_sum(6, [2, 3, 8, 11, -1])
| False
```

a) `len(L)`

b) `sum(L)`

c) `x`

d) $2^{\texttt{len(L)}}$

e) `len(L)`

f) `len(L)*sum(L)`

g) Don't know

# Subset sum using dynamic programming

**subset_sum_dp.py**

```python
def subset_sum_solution(x, L):
    @memoize
    def solve(value, k):
        if k == 0:
            if value == 0:
                return []
            else:
                return None
        solution = solve(value, k - 1)
        if solution != None:
            return solution
        solution = solve(value - L[k - 1], k - 1)
        if solution != None:
            return solution + [L[k - 1]]
        return None

    return solve(x, len(L))
```

**Python shell**

```
> subset_sum_solution(11, [2, 3, 8, 11, -1])
| [3, 8]
> subset_sum_solution(6, [2, 3, 8, 11, -1])
| None
```

# Knapsack problem

| | Volume | Value |
|---|---|---|
| 0 | 3 | 6 |
| 1 | 3 | 7 |
| 2 | 2 | 8 |
| 3 | 5 | 9 |

- Given a **knapsack** with volume **capacity C**, and set of **objects** with different **volumes and value**
- **Objective**: Find a subset of the objects that fits in the knapsack (sum of volume ≤ capacity) and has maximal value
- **Example**: If C = 5 and the volume and weights are given by the table, then the maximal value 15 can be achieved by the 2nd and 3rd object
- Let V(c, k) denote the maximum value achievable by a subset of the first k objects within capacity c

$$V(c, k) = \begin{cases} 0 & \text{if } k = 0 \\ V(c, k-1) & \text{volume}[k-1] > c \\ \max\{V(c, k-1), \text{value}[k-1] + V(c - \text{volume}[k-1], k-1)\} & \text{otherwise} \end{cases}$$

# Knapsack – maximum value

knapsack.py

```python
def knapsack_value(volume, value, capacity):
    @memoize
    def solve(c, k):  # solve with capacity c and objects 0..k-1
        if k == 0:  # no objects to put in knapsack
            return 0
        v = solve(c, k - 1)  # try without object k-1
        if volume[k - 1] <= c:  # try also with object k-1 if space
            v = max(v, value[k - 1] + solve(c - volume[k - 1], k - 1))
        return v

    return solve(capacity, len(volume))
```

Python shell

```
> volumes = [3, 3, 2, 5]
> values = [6, 7, 8, 9]
> knapsack_value(volumes, values, 5)
| 15
```

# Knapsack – maximum value and objects

**knapsack.py**

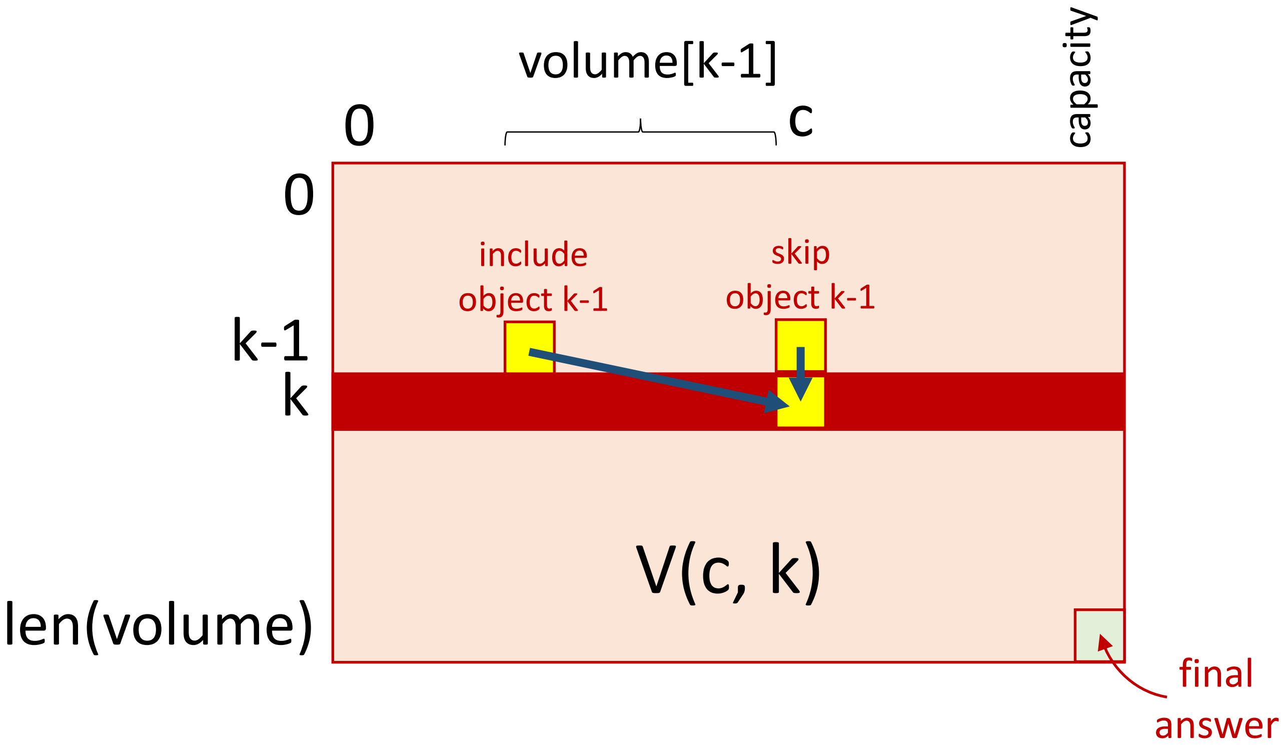
```python
def knapsack(volume, value, capacity):
    @memoize
    def solve(c, k):  # solve with capacity c and objects 0..k-1
        if k == 0:  # no objects to put in knapsack
            return 0, []
        v, solution = solve(c, k - 1)  # try without object k-1
        if volume[k - 1] <= c:  # try also with object k-1 if space
            v2, sol2 = solve(c - volume[k - 1], k - 1)
            v2 = v2 + value[k - 1]
            if v2 > v:
                v = v2
                solution = sol2 + [k - 1]
        return v, solution

    return solve(capacity, len(volume))
```

**Python shell**

```
> volumes = [3, 3, 2, 5]
> values = [6, 7, 8, 9]
> knapsack(volumes, values, 5)
| (15, [1, 2])
```

# Knapsack - Table

$$V(c, k) = \begin{cases} 0 & \text{if } k = 0 \\ V(c, k-1) & \text{value}[k-1] > c \\ \max\{V(c, k-1), \text{value}[k-1] + V(c - \text{volume}[k-1], k-1)\} & \text{otherwise} \end{cases}$$




- systematic fill out table
- only need to remember two rows

# Knapsack – Systematic table fill out

**knapsack_systematic.py**

```
def knapsack(volume, value, capacity):
  ① solutions = [(0, [])] * (capacity + 1)
  ② for obj in range(len(volume)): ⑤
         for c in reversed(range(volume[obj], capacity + 1)): ④
             prev_v, prev_solution = solutions[c - volume[obj]]
             v = value[obj] + prev_v
             if solutions[c][0] < v:
              ③ solutions[c] = v, prev_solution + [obj]

     return solutions[capacity]
```

**Python shell**

```
> volumes = [3, 3, 2, 5]
> values = [6, 7, 8, 9]
> knapsack(volumes, values, 5)
| (15, [1, 2])
```

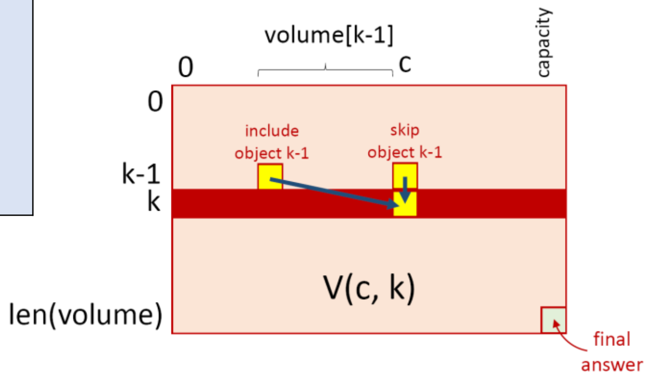
① base case k = 0

② consider each object

③ `solutions[c:]` current row
`solutions[:c]` previous row

④ compute next row right-to-left

⑤ `solutions[:volume[obj]]`
unchanged from previous row



# Summary

- Dynamic programming is a general approach for recursive problems where one tries to avoid recomputing the same expresions repeatedly

- **Solution 1: Memoization**
  - add dictionary to function to remember previous results
  - decorate with a @memoize decorator
- **Solution 2: Systematic table fill out**
  - can need to compute more values than when using memoization
  - can discard results not needed any longer (reduced memory usage)

# Coding competitions and online judges

If you like to practice your coding skills, there are many online "judges" with numerous exercises and where you can upload and test your solutions.

- Project Euler
- Kattis
- CodeForces
- Topcoder

See cs.au.dk/~gerth/code/